Mirror Symmetry, D-branes and Counting Holomorphic Discs

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Abstract

We consider a class of special Lagrangian subspaces of Calabi-Yau manifolds and identify their mirrors, using the recent derivation of mirror symmetry, as certain holomorphic varieties of the mirror geometry. This transforms the counting of holomorphic disc instantons ending on the Lagrangian submanifold to the classical Abel-Jacobi map on the mirror. We recover some results already anticipated as well as obtain some highly non-trivial new predictions.

1. Introduction

Calabi-Yau geometry has been the source of many interesting physical insights in string theory. A key role is played by mirror symmetry which relates questions involving the Kahler geometry of the Calabi-Yau to complex geometry of a mirror Calabi-Yau (or more generally complex parameters characterizing a mirror description of the N=2 worldsheet theory). A simple proof of mirror symmetry has appeared in [1] based on enlarging the gauge system of the linear sigma model [2] and applying T-duality to the charged fields of the theory. It is thus natural to ask how this acts on the D-branes. It is expected that even and odd branes of the two geometries are exchanged under the mirror symmetry. This maps Lagrangian submanifolds (which are half the dimension of the Calabi-Yau) on one side to the complex submanifolds of the mirror geometry. Aspects of this action were studied for certain massive sigma models in [3]. The aim of this paper is to extend mirror symmetry to certain special Lagrangian submanifolds of Calabi-Yau and its mirror complex geometry. As a by-product we are able to count the holomorphic discs ending on the Lagrangian submanifolds using the Abel-Jacobi map of the mirror manifold.

The organization of this paper is as follows: In section 2 we discuss aspects of toric geometry with emphasis on certain special Lagrangian submanifolds associated to it. These constructions have already appeared in the mathematics literature [4][5] and they are very natural from the viewpoint of linear sigma models. In section 3 we discuss mirror symmetry, as derived in [1], and apply it to the Lagrangian submanifolds discussed in section 2 to obtain holomorphic submanifolds of the mirror geometry. In section 4 we show how the holomorphic disc amplitudes of the A-model in certain cases are related to Abel-Jacobi map of the mirror geometry. We use this result in section 5 to compute some holomorphic disc instanton corrections. In particular we confirm the result of [6] which predicts a universal $1/n^2$ multi-covering formula for disc instantons. We also find highly non-trivial predictions for the number of holomorphic disc instantons in various situations which pass the integrality check of [6].

2. Toric Geometry and Special Lagrangian Submanifolds

We begin this section by briefly reviewing certain aspects of toric geometry. Let $X = \mathbb{C}^n$ be parameterized by x^1, \dots, x^n , and endowed with flat Kahler form $\omega = i \sum_i dx^i \wedge d\bar{x}^i$. We can also view ω as

$$\omega = \sum_{i} d|x^{i}|^{2} \wedge d\theta^{i} \tag{2.1}$$

where θ^i denotes the angular variable in the x^i plane.

Consider a Lagrangian submanifold $L = \mathbb{R}^n$ of \mathbb{C}^n corresponding to fixed θ^i . This n-dimensional real space is parametrized by $|x^i|^2$ and the fact that it is Lagrangian follows trivially as θ^i are constants, so ω vanishes on it. Of course this description is valid as long as we are away from loci where any $x^i = 0$. Note that \mathbb{C}^n can be viewed as a \mathbb{C}^n torus fibration over L, where the fibration degenerates at the boundaries of L (where any $|x^i|^2 = 0$). This is the basic setup of toric geometry.

We can now describe other Lagrangian submanifolds of \mathbb{C}^n . Consider any submanifold $D^r \in L$ of dimension $r \leq n$. For each point $p \in D^r$ consider the r dimensional tangent space $T_p(D^r) \hookrightarrow T_p(L)$. This defines an n-r dimensional subspace of the fiber T^n over that point, orthogonal with respect to ω to the tangent directions to D^r . If the slope of the subpace D^r is rational then the corresponding n-r dimensional subpace of T^n is a torus $T^{n-r} \subset T^n$ over p. Let us assume that D^r has rational slope at all points—this effectively reduces one to rational linear subspaces of L. In this way we obtain a Lagrangian submanifold associated to each such subspace. We can characterize a linear rational subspace of L by n-r sets of n-tuple integers q_i^{α} , where i=1,...,n and $\alpha=1,...,n-r$ such that

$$\sum_{i} q_i^{\alpha} |x^i|^2 = c^{\alpha} \tag{2.2}$$

where c^{α} are constants (not necessarily integers). One can also write these in terms of r vectors v_{β} as

$$|x^i|^2 = v^i_\beta s^\beta + d^i$$

where β runs from $1, ..., r, d^i$ are constants and

$$q^{\alpha} \cdot v_{\beta} = \sum_{i=1}^{n} q_i^{\alpha} v_{\beta}^i = 0.$$

Note that the constraints on the θ^i are

$$\sum_{i} v_{\beta}^{i} \theta^{i} = 0 \tag{2.3}$$

and equivalently

$$\theta^i = q_i^{\alpha} \phi_{\alpha}.$$

Below, we will be interested in a subset of such Lagrangian submanifolds known as special Lagrangian submanifolds, which satisfy the property that for each α

$$\sum_{i=1}^{n} q_i^{\alpha} = 0. {(2.4)}$$

So far we have ignored the discussion of boundaries of L and the other Lagrangian submanifolds, and whether the above constructions can be extended to true Lagrangian submanifold without boundary. L itself is not Lagrangian but it will be if we take 2^n fold cover of it (by choosing, for each i, both the $\theta^i = 0$ section and $\theta^i = \pi$ section of \mathbb{C}^n) and it will correspond to the real subspace of \mathbb{C}^n .

Similar statement holds for subspaces $D^r \subset L$ with boundaries at $x^i = 0$ for some of the i. But also sometimes it is not necessary to do this doubling. Consider for example \mathbb{C}^2 in which case L can be identified with the positive quadrant of the 2 dimensional plane. Consider the Lagrangian submanifold $D \subset L$ given by

$$(q_1, q_2) = (1, -1).$$

This corresponds to the subspace

$$|x^1|^2 - |x^2|^2 = c,$$
 $\theta^1 + \theta^2 = 0.$

For generic c>0 where $|x^2|^2=0$ this meets the boundary of L at $|x^1|^2=c$ and unless we double the geometry, D will give rise to a Lagrangian submanifold with boundary. However if we consider the particular case where c=0 (see figure 1) then we do not need to double the geometry and D corresponds to a Lagrangian submanifold without any boundaries. It in fact corresponds to $x^1=\overline{x}^2$.

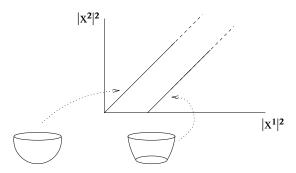


Fig.1: Lagrangian submanifolds of \mathbb{C}^2 with and without boundaries, projected to the two dimensional base $L = (|x^1|^2, |x^2|^2)$.

2.1. Calabi-Yau Geometry and Special Lagrangian Submanifolds

So far we have discussed a very simple Calabi-Yau geometry, namely the non-compact \mathbb{C}^n . However, toric geometry is also very useful in describing rather non-trivial Calabi-Yau manifolds, both as non-compact weighted projective spaces or complete intersection in products of weighted projective spaces. We first review some aspects of these constructions and their relation to linear sigma model.

Start again with $X = \mathbb{C}^n$ as a torus T^n fibration over L. The torus acts on X by phase rotations $x^i \to e^{i\theta^i}x^i$ and this action preserves the Kahler form. Naive quotients by subgroups of $U(1)^n$ are neither smooth nor Kahler (or complex for that matter) but there is a well known prescription that circumvents both problems.

Pick a $G = U(1)^{n-k}$ subgroup of the isometry group acting on X by

$$x^i \to e^{iQ_i^a \epsilon_a} x^i \tag{2.5}$$

for some choice of charges Q^a . If we define the quotient Y = X//G to be obtained by setting for a = 1, ..., n - k

$$\sum_{i} Q_i^a |x^i|^2 = r^a \tag{2.6}$$

on X and dividing the resulting space by G than the quotient manifold Y is a complex, Kahler manifold. This definition has a natural realization through linear sigma models [2] where one considers a two-dimensional N=2 gauge theory with gauge group $G=U(1)^{n-k}$ and n fields Φ^i which have charges Q_i^a under the corresponding U(1)'s. The above constraint (2.6) is the minimum of the D-term potential $D^a=0$ and modding out the resulting space by G is considering the gauge inequivalent orbits of the vacuum.

For sufficiently generic choices of parameters r^a , G acts freely on (2.6) and Y is a smooth manifold. The Kahler form ω_Y on the quotient is obtained from the Kahler form ω on X by restricting to $D^a = 0$ subspace and dividing by G.

Y can be also be viewed as a (generalization of) weighted projective space $Y = X/G^{\mathbf{C}}$, where instead of setting D-terms to zero and dividing by G we take an ordinary quotient by the complexified gauge group $G^{\mathbf{C}}$

$$x^i \sim \prod_a \left(\lambda_a\right)^{Q_i^a} x^i$$

for λ_a in \mathbb{C}^* , and with suitable subspaces of X deleted. The manifold Y is in addition to being Kahler, a non-compact Calabi-Yau space if, for each a,

$$\sum_{i=1}^{n} Q_i^a = 0. (2.7)$$

Note that this requires having some negative charges Q_i^a and the corresponding fields lead to the non-compact directions of the Calabi-Yau. Under the above condition the holomorphic n-form $\Omega = dx^1 \wedge \ldots \wedge dx^n$ is $G^{\mathbf{C}}$ invariant and descends to a holomorphic k form on Y by contraction with n-k generators of the complexified gauge group, $\Omega_Y = i_{q^1} \ldots i_{q^{n-k}} \Omega$.

We have to clarify what we mean by the manifold Y being a Calabi-Yau space: It has a trivial canonical line bundle. This does not mean that the metric induced from its embedding in X agrees with the Ricci-flat Calabi-Yau metric. In fact it does not. However as discussed in [2] the linear sigma model with Kahler form induced from X is a quantum theory on the worldsheet which flows in the infrared to a conformal theory with an approximately Ricci-flat metric (note that generally the metric picked by the conformal theory is a refinement of the Ricci-flat metric on the CY which only at large radii becomes the Ricci-flat metric). The RG flow affects the D-term and leaves the superpotential terms unchanged – which is why for issues of topological strings, mirror symmetry works equally well for this non-Ricci-flat induced metric.

We now turn to construction of Lagrangian submanifolds of Y, which can be defined since Y is Kahler. First, note that the geometric picture with X realized as a T^n fibration over L descends to the quotient space. The manifold Y is a $T^n/G = T^k$ fibration over restriction of $L = R^n$ to subspace (2.6) determined by charges Q_a . The restriction, which we will denote by L_Y , is clearly Lagrangian in the induced Kahler form ω_Y . In fact all the Lagrangian submanifolds of X we constructed in the previous section descend to Lagrangian submanifolds of Y. Because the Kahler form on Y derives from the one on X by restriction modulo G Lagrangian submanifolds on X, provided they make sense in the quotient, are automatically Lagrangian on Y as well.

The condition we need to impose is that the v^i_{β} should lead to gauge invariant constraints in (2.3), and this means that

$$Q^a \cdot v_\beta = \sum_{i=1}^n Q_i^a v_\beta^i = 0.$$

The gauging constrains v_{β}^{i} but it does *not* put a constraint on the Lagrangian charges q_{i}^{α} . The q's and Q's, up to taking linear combinations, are thus the data specifying homology class of Lagrangian submanifold of Y.

Again note that the same comment made above about the exact metrics on the Calabi-Yau manifold applies equally well to the Lagrangian subspaces. Namely the Lagrangian submanifolds we have constructed here will not necessarily be Lagrangian with respect to the exact metric picked by the conformal theory. However one expects that the Lagrangian submanifold gets deformed in the IR, just as the metric gets deformed, so as to continue to be Lagrangian. Again, as far as the issues of topological strings are concerned these are D-term variations which do not affect the topological computations.

Given a Calabi-Yau manifold, one can formulate the condition for Lagrangian submanifold to be of minimal volume in terms of the holomorphic n-form Ω . One defines a special Lagrangian cycle to be that on which Ω has constant phase [4][7]. If the Lagrangian submanifold satisfies this, it is volume minimizing in its homology class. In our case, Y is Calabi-Yau if $\sum_i Q_i^a = 0$. Since all the Lagrangian submanifolds we constructed correspond to planar subspaces D^r of L_Y phase of Ω_Y on each is given by $\sum_i \theta^i$, so for our constructions to lead to special Lagrangians this sum must be constant. In order for the special Lagrangian condition to be satisfied on D^r without over-constraining the Lagrangian, we must have that of one the v^β is $v_i^\beta = (1, 1, 1, ..., 1)$. This in turn, by virtue of $q^\alpha \cdot v_\beta = 0$, implies the constraint we stated before:

$$\sum_{i} q_i^{\alpha} = 0 \tag{2.8}$$

for all α . From now on, we restrict our attention to Lagrangian submanifolds which satisfy this.

We can also impose hypersurface constraints in Y or consider complete intersections in the weighted projective spaces. Physically this corresponds to deforming the action of the two dimensional sigma model by certain superpotential terms [2]. In these cases, the restriction of the Kahler form of Y to the corresponding subspaces gives a Kahler structure to the Calabi-Yau. Thus the intersection of the Lagrangian submanifolds we have constructed with the Calabi-Yau manifold, continue to be Lagrangian.

To summarize we have constructed, for non-compact or compact Calabi-Yau, characterized by charges Q_i^a of the fields Φ_i of the linear sigma model, Lagrangian submanifolds characterized by "charges" q_i^a . These are special Lagrangian if and only if (2.8) holds. These subspaces are Lagrangian relative to the induced Kahler form from their embedding in \mathbb{C}^n . Moreover, they are expected to flow in the IR to special Lagrangian submanifolds relative to the Kahler form corresponding to the metric which gives rise to a conformal theory on the worldsheet (and which at very large radii is close to the Calabi-Yau metric). For the sake of a shorter terminology when we consider D-branes wrapped around such special Lagrangian submanifolds we will refer to them as "A-branes" (as they preserve the A-model topological charge).

3. Mirror Symmetry Action on Lagrangian D-branes

In this section we obtain the mirror of the Lagrangian D-branes constructed in section 2. We first review the derivation of mirror symmetry [1] and then use it to find the "B-branes" that are mirror of the "A-branes". We will mainly concentrate on the Calabi-Yau case, and D-branes wrapped over the special Lagrangian submanifolds—however many of our remarks apply to more general settings including the non-Calabi-Yau cases.

Consider, for definiteness, a linear sigma model with fields (Φ_i, P) where i = 1, ..., n charged under a U(1) with charges given by $(Q_i, Q)^{-1}$. The Calabi-Yau condition (equivalently the vanishing of the beta function) requires

$$Q + \sum_{i} Q_i = 0$$

which implies that at least some of the charges are negative. Let us suppose that Q < 0. The above equation is equivalent then to

$$|Q| = \sum_{i} Q_{i}$$

There is a potential in the linear sigma model which comes from the D-term, and the minimum of this potential is given by

$$\sum Q_i |\phi_i|^2 + Q|P|^2 = r. {(3.1)}$$

¹ In this section for convenience we have shifted our notation from the previous section in that we have n + 1 total fields rather than the n fields of the previous section.

The r parameter is a FI term which combines with the U(1) θ angle to give a complexified Kahler parameter $t = r + i\theta$. When r > 0 the geometry of this minimum modulo gauge transformation can be viewed as a non-compact weighted projective space with weights given by (Q_i, Q) . The Kahler class of the compact part of the space depends linearly on r, and the non-compact direction is parameterized by the field P.

To obtain the mirror model we follow [1] and introduce dual (twisted) chiral fields Y_i such that

$$ReY_i = |\Phi_i|^2$$

$$ReY_P = |P|^2.$$
(3.2)

This is obtained by acting with T-duality on all of the n+1 fields of the original theory.² It is also convenient to define

$$y_i = \exp(-Y_i), \qquad y_P = \exp(-Y_P)$$

and this is natural given the fact that the imaginary part of Y_i are periodic variables, of period 2π . Moreover the mirror version of the equation (3.1) is given by

$$y_p^Q \prod y_i^{Q_i} = e^{-t} \to \prod y_i^{Q_i} = e^{-t} y_p^{|Q|}.$$
 (3.3)

The mirror theory is a Landau-Ginsburg theory in terms of Y_i , P with a superpotential

$$W = \sum_{i} y_i + y_p$$

subject to (3.3). For simplicity, let us assume that all Q_i divide |Q| and put $m_i = |Q|/Q_i$. We then can solve (3.3) by introducing new fields $\tilde{y}_i^{m_i} = y_i$ in terms of which we have

$$W = F(\tilde{y}_i) = \sum_i \tilde{y}_i^{m_i} + e^{t/|Q|} \prod_i \tilde{y}_i.$$

² The proposal for studying the geometry of the mirror Calabi-Yau in terms of mirror symmetry action on tori [7] also uses T-duality, but in a different set up. For example for the case of quintic the approach of [1] applies T-duality to 6 fields, whereas in the [7] approach one applies it to 3 fields. The approach of [7] is related to the heuristic derivation of Batyrev's proposals for mirror pairs given in [8]. For some recent progress in this direction see for example [9]. However all approaches to understanding mirror symmetry have the common feature of using T-duality in one way or another.

To be precise, for the new fields to be well defined functions of the old, we have to consider an orbifold of this acting on \tilde{y}_i by all m_i 'th roots of unity which leave $\prod \tilde{y}_i$ invariant.

Mirror symmetry above can also be stated in the geometric language. We first recall the compact Calabi-Yau case. The original theory reduces to a compact Calabi-Yau sigma model if we add a gauge invariant superpotential $PG(\phi_i)$. As discussed in [1] this does not affect the LG superpotential W above, except to make the fundamental fields of the theory to be y_i instead of the Y_i . Then, the LG theory is equivalent to an orbifold of the hypersurface

$$F(\tilde{y}_i) = 0$$

in the corresponding weighted projective space. This can be written in a coordinate patch where, say $\tilde{y}_n \neq 0$ as

$$F(\widehat{y}_i, \widehat{y}_n = 1) = 0$$

in inhomogeneous variables $\hat{y}_i = \tilde{y}_i/\tilde{y}_n$.

If in the original theory we do not add a superpotential $PG(\Phi_i)$, then the A-model continues to correspond to a non-compact Calabi-Yau space. In this case the mirror theory is geometrically equivalent [3] to a non-compact Calabi-Yau

$$xz = F(\tilde{y}_i) \tag{3.4}$$

where x, z are C-valued and $\tilde{y}_i \in \mathbb{C}^*$ (i.e. in this case the Y_i are the good variables). There is still a \mathbb{C}^* action on the above space, which allows us to set one of the \tilde{y}_i to 1 (which one we set to one, depends on the patch we wish to study the mirror geometry in). Note that the non-compact case has two dimensions more compared to the compact case (given by the extra variables x, z) but both the compact and the non-compact geometry are characterized by F.

To avoid unnecessary complication in notation, in the following we will drop the tilde off of $\tilde{y}_i \to y_i$. Generalization of the above discussion to multiple U(1)'s is straightforward and can be found in [1].

3.1. Identification of the B-branes

The mirror of Lagrangian submanifolds are expected to be holomorphic submanifolds of the mirror, which we call B-branes. Note that the action of T-duality on \mathbb{C}^n already suggests that the mirror of D^r , whose fiber is a $T^{n-r} \subset T^n$, is a T^r fibration over D^r i.e. it should be specified by n-r complex equations. This we will indeed find to be the case.

In the discussion of special Lagrangian submanifolds we noted that they are characterized by certain "charges" q_i^{α} . These in particular restrict the Φ_i by

$$\sum_{i} q_i^{\alpha} |\Phi_i|^2 = c^{\alpha}$$

with no loss of generality we have assumed that $q_P^{\alpha} = 0$ (we can use the (3.1) constraint to write the equations without P). Note also that the condition of being special Lagrangian submanifold implies that $\sum_i q_i^{\alpha} = 0$. Given the discussion above, it is easy to write the mirror of the above Lagrangian. Namely, from (3.2) and from the fact that we expect a holomorphic equation we immediately find that

$$\prod_{i=1}^{n-1} y_i^{q_i^{\alpha} \cdot m_i} = \epsilon^{\alpha} \exp(-c^{\alpha}), \tag{3.5}$$

where e^{α} is a phase, which can be combined with e^{α} to give it an imaginary part. This implies that in the compact case in addition to $F(y_i) = 0$ we consider the holomorphic subspace given by the above equations. For the non-compact case the same holds, but for the subspace of $xz = F(y_i)$.

Below we give some examples of the mirror action on the A-branes leading to B-branes on the mirror manifold for both compact and non-compact cases.

3.2. Compact Examples

Consider the quintic three-fold as an example. The field content of the linear sigma model is a U(1) gauge theory with six fields with charges

$$(\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, P) = (1, 1, 1, 1, 1, -5)$$

(together with a superpotential defining the complex structure of the quintic). The mirror theory is given by

$$[y_1^5 + y_2^5 + y_3^5 + y_4^5 + y_5^5 + e^{t/5}y_1y_2y_3y_4y_5 = 0]/\Gamma$$
(3.6)

in CP^4 , where we consider a $\Gamma = Z_5^3$ orbifold of it given by multiplication of each y_i by a fifth root of unity, preserving $y_1y_2y_3y_4y_5$.

Now consider the mirror of the Lagrangian submanifold defined by the charge q_i^1 given by

$$q^1 = (1, -1, 0, 0, 0, 0).$$

This means that the Lagrangian submanifold satisfies

$$|\Phi_1|^2 - |\Phi_2|^2 = c^1.$$

Then according to (3.5) the mirror is given by the subspace of (3.6) satisfying

$$y_1^5 = y_2^5 \exp(-c^1). (3.7)$$

This is a two complex dimensional holomorphic subspace.

As another example, consider the Lagrangian submanifold given by two charges q^1, q^2 with q^1 as given above, and

$$q^2 = (0, 0, 1, 0, 0, -1)$$

which means that we are imposing that the Lagrangian submanifold intersects the base at

$$|\Phi_3|^2 - |P|^2 = c^2.$$

As mentioned before, we can change this (by imposing the condition of the D-terms)

$$|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 + |\Phi_4|^2 + |\Phi_5|^2 - 5|P|^2 = r$$

to

$$-|\Phi_1|^2 - |\Phi_2|^2 + 4|\Phi_3|^2 - |\Phi_4|^2 - |\Phi_5|^2 = 5c^2 - r = \tilde{c}^2$$

where we have introduced \tilde{c}^2 for convenience. In other words this is effectively equivalent to taking $q^2 = (-1, -1, 4, -1, -1, 0)$. This leads to the mirror brane given as the locus characterized, in addition to the constraint (3.7) by

$$(y_3^5)^4 = y_1^5 y_2^5 y_4^5 y_5^5 e^{-c_2}$$

which is a complex dimension one subvariety of the mirror to quintic.

3.3. Non-compact Examples

As our first non-compact example we consider the geometry given by the O(-1)+O(-1) bundle over \mathbf{P}^1 . This is described by a U(1) linear sigma model with four fields with charges

$$(\Phi_1, \Phi_2, \Phi_3, \Phi_4) = (1, 1, -1, -1).$$

The mirror of this theory is given by the geometry

$$xz = y_2 + y_3 + y_4 + e^{-t} \frac{y_3 y_4}{y_2}$$
(3.8)

where $x, z \in \mathbf{C}$ and y_2, y_3, y_4 are \mathbf{C}^* variables, and we have to go to a patch where one of the $y_i = 1$ (we have eliminated y_1 from the superpotential by the equation $y_1y_2 = y_3y_4e^{-t}$, as we will be in a regime of parameters where y_1 is small and varies little). The convenient choice of patch for the A-branes we will consider turns out to be given by $y_4 = 1$, in which case the equation of the mirror is

$$xz = y_2 + y_3 + 1 + e^{-t} \frac{y_3}{y_2}. (3.9)$$

We consider the A-brane characterized by two charges

$$q^1 = (0, 1, 0, -1)$$

$$q^2 = (0, 0, 1, -1)$$

which corresponds to the projection on the base given by

$$|\Phi_2|^2 - |\Phi_4|^2 = c_1$$

$$|\Phi_3|^2 - |\Phi_4|^2 = c_2$$

and consequently $|\Phi_1|^2 - |\Phi_4|^2 = r - c_1 + c_2$. This makes sense for generic c_1, c_2 see figure 2^3 . However, as noted before, there are certain codimension one loci in parameter space where something special happens: The generic Lagrangian submanifold, corresponding to case III in figure 2, degenerates to two submanifolds and we can in principle wrap the D-brane over any one of them. This happens for example if

 $[\]overline{}^3$ For a geometric meaning of such figures as well as an interpretation in terms of branes see [8] .

(I) $r > c_1 > 0$ and $c_2 = 0$ or if (II) $c_1 = 0$ and $c_2 > 0$ (see figure 2). For either of these two cases the D-brane will *not* have a deformation away from this special locus, as it would acquire a boundary. Precisely for these branes we will later compute a non-vanishing superpotential using the mirror B-brane.

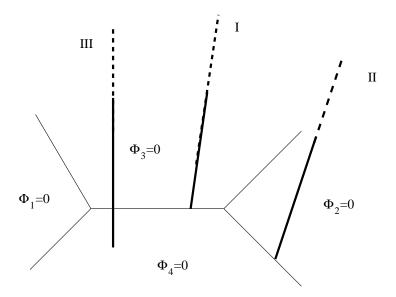


Fig.2: The projection of the Lagrangian submanifold on the base corresponds to a straight line (III). For special values of c_1, c_2 the line will intersect the loci with a pair of vanishing circles. This can happen in two inequivalent ways. For $r > c_1 > 0$, $c_2 = 0$ it ends on the interval $\Phi_3 = \Phi_4 = 0$ (I) and if $c_1 = 0$ it ends on the line $\Phi_4 = \Phi_2 = 0$ (II).

The mirror for the general values of c_1, c_2 is given by the subspace of (3.9)

$$y_2 = e^{-c_1} y_4 \qquad y_3 = e^{-c_2} y_4.$$

This implies that in the $y_4 = 1$ patch using (3.9), we look at the subspace

$$y_2 = e^{-c_1}$$
 $y_3 = e^{-c_2}$ of $xz = 1 + y_2 + y_3 + e^{-t} \frac{y_3}{y_2}$. (3.10)

Note that this subspace is given by a one-dimensional complex B-brane characterized by

$$xz = \text{const.}$$

Note that if the constant on the RHS is zero, then the B-brane splits to two B-branes given by x = 0 or z = 0. This is the mirror of the statement we made

about the A-brane. Let us check this for the two cases mentioned above in the large radius limit, where the two pictures should match.

Consider first the case I where $c_2 = 0$ and where we consider the large radius limit r >> 0 and where c_1 is large but less than r/2 (i.e. when the A-brane intersects the \mathbf{P}^1 near the equator and towards the south pole). In this limit the RHS is dominated by $1 + y_3$ and if we take the imaginary part of c_2 (which was not fixed by the mirror map) to be $i\pi$ we see that for this brane $y_3 = -1$ and the RHS vanishes. Thus the mirror of the half A-brane agrees in this limit with the locus where xz = 0 as expected. The generalization of this condition is predicted by the mirror map to be choosing y_3 as a function of y_2 such that the RHS vanishes away from the large radius limit. Writing in terms of the \mathbf{C}^* variables $y_2 = e^u$ and $y_3 = e^v$, this means that we can determine v as a function of u such that F(u, v) = 0 where

$$xz = F(u, v) = 1 + e^{u} + e^{v} + e^{-t}e^{(v-u)}$$
.

To leading order $v = i\pi$, but more generally we have that

$$v = i\pi + \log \frac{1 + e^u}{1 + e^{-t - u}},\tag{3.11}$$

as implied by F(u, v) = 0. Note that here u geometrically denotes the size of the disc in the \mathbf{P}^1 which ends on the brane. This is the sense, as we will discuss later, in which u is the "good variable" from the viewpoint of topological string.

In the case (II) where we consider $c_1 = 0$ and $c_2 > 0$, in the large radius limit we have $e^u = -1$, $e^v \to 0$ and again the RHS of the equation xy = F(u, v) vanishes. More generally, i.e. away from the large radius limit, to obtain the mirror of the single brane we demand vanishing of F which in this case gives

$$u = i\pi + \log\left[\frac{1+e^{v}}{2} + \frac{1}{2}\sqrt{(1+e^{v})^{2} - 4e^{-t+v}}\right].$$
 (3.12)

Note that in this case v is the good variable, as it measures the size of the disc passing through the south pole of \mathbf{P}^1 .

For another example, consider the local model given by a non-compact Calabi-Yau containing a $\mathbf{P}^1 \times \mathbf{P}^1$. This can be realized with a $U(1)^2$ gauge group with five matter fields, with charges

$$Q^1 = (1, 1, 0, 0, -2)$$

$$Q^2 = (0, 0, 1, 1, -2)$$

the mirror manifold in the $y_5 = 1$ patch is given by

$$xz = y_1 + \frac{e^{-t_1}}{y_1} + y_3 + \frac{e^{-t_2}}{y_3} + 1. (3.13)$$

We consider the Lagrangian submanifold given by

$$q^1 = (1, 0, 0, 0, -1)$$

$$q^2 = (0, 0, 1, 0, -1),$$

which means that we have put

$$|\Phi_1|^2 = |\Phi_5|^2 + c_1$$

$$|\Phi_3|^3 = |\Phi_5|^2 + c_2.$$

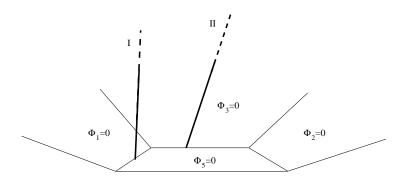


Fig.3: Some special A-branes in $\mathbb{P}^1 \times \mathbb{P}^1$.

The mirror is given by $y_1 = e^{-c_1}y_5$ and $y_3 = e^{-c_2}y_5$, or going to the $y_5 = 1$ patch, by $y_1 = e^{-c_1}$ and $y_3 = e^{-c_2}$ subspace of (3.13). If we substitute $y_1 = e^u$, $y_3 = e^v$ into (3.13), we get an equation of form xz = F(u, v). The condition that the brane splits to two parts is again the condition that is quantum corrected to F(u, v) = 0. For example in the large radius limit if we consider $0 << c_1 < r/2$ and $c_2 = 0$ we have the brane II depicted in figure 3. The mirror brane is $e^u = e^{-c_1} \to 0$, $e^v = -1$ (by a suitable choice of imaginary part of c_2) and so F = 0 is satisfied. More generally, we have v determined in terms of u from F(u, v) = 0.

4. Topological Strings and Superpotentials

In the previous sections we have considered certain special Lagrangian submanifolds in Calabi-Yau manifolds and their mirrors, the holomorphic submanifolds of the mirror geometry. This statement descends to the topological subsector of these theories. In particular, topological A-models admit Lagrangian D-branes (which is why we called them A-branes) and topological B-models admit holomorphic D-branes (and hence the terminology B-branes) [10]. Since mirror symmetry converts the A-type topological string to B-type topological string, and the A-branes to B-branes, it is natural to ask how one can use mirror symmetry to compute A-type topological string invariants in terms of the B-model. This general setup and its consequences for topological strings has been discussed in [11].

The A-model topological string amplitudes are given in terms of the enumerative geometry having to do with holomorphic maps from Riemann surfaces with boundaries to a target Calabi-Yau manifold where the boundary ends on a Lagrangian D-brane⁴. This in general involves a complicated enumerative geometry question and there is no direct approach known to computing it. Using the large N duality conjecture [12] there have been some cases where one can compute certain corrections involving holomorphic maps from Riemann surfaces with boundaries to target space geometry [6][13][14]. Moreover based on what the topological strings compute in the context of type II superstrings certain integrality properties for the A-model amplitude can be predicted [6] generalizing those without D-branes [15]. For example it is shown that the disc amplitudes in the A-model will have the general structure given by

$$F_{Disc} = W = \sum_{n=0}^{\infty} \sum_{\vec{m}, \vec{k}} \frac{d_{\vec{k}, \vec{m}}}{n^2} q^{n\vec{k}} y^{n\vec{m}}, \tag{4.1}$$

where $q = e^{-t}$ for t a basis for complexified closed string Kahler classes, and y related by exponentiation to the complexified open string Kahler class which measures the volumes of holomorphic discs. The integers $d_{\vec{k},\vec{m}}$ in the above formula count "primitive disc instantons" in relative homology class (\vec{m}, \vec{k}) ,

⁴ The degenerate limit of such maps gives rise to ordinary Chern-Simons theory on the Lagrangian submanifold [10].

where \vec{m} labels the class on the boundary (i.e. an element of of H_1 of the brane) and \vec{k} labels an H_2 element of the Calabi-Yau, and the sum over n sums the multi-coverings of these. The reason we have denoted the disc amplitude also by W is that in the context of type II superstrings, if we consider the branes filling the space-time (which only makes sense if the Calabi-Yau is non-compact, for the brane flux to have somewhere to go) the topological string disc amplitude has the interpretation of superpotential corrections to 4d N=1 supersymmetric theory [16][17][6][18][19]. Note that the above form of W makes sense only in the large radius limit and that this structure requires W to have very strong integrality properties.

On the B-model side the topological string is related to holomorphic Chern-Simons theory [10] if we consider the D-brane wrapped over the entire Calabi-Yau, or its dimensional reductions depending on the dimension of the D-brane (as we will discuss below). Thus the hope is to map the difficult problem of computations on the A-model side to some easy computations on the B-model side. For example, if we consider an annulus, then the B-model partition function is given by a holomorphic Ray-Singer torsion and this would compute, by mirror symmetry holomorphic maps from the annulus to the original Calabi-Yau geometry with the boundaries of the annulus ending on the mirror Lagrangian submanifolds [11]. Similarly higher genus Riemann surfaces with boundaries have interpretation in terms of the holomorphic Chern-Simons theory coupled to the bulk complex structure (the Kodaira-Spencer theory [20]).

The disc amplitude computes the classical action on the B-model side, which as noted above corresponds to the holomorphic Chern-Simons action or its reductions on the worldvolume of the B-brane. Thus by computing the classical action on the B-model side, we can compute the A-model holomorphic disc instantons. We will use this idea to compute, using mirror symmetry, the A-model disc instanton corrections.

4.1. B-model Computation of Superpotential for a 2-brane

Consider a Calabi-Yau manifold Y in the context of topological B-model. If we have a 6-brane wrapping the entire Y, which can be viewed as introducing an open string sector with purely Neumann boundary conditions on Y, we obtain a holomorphic Chern-Simons gauge system living on the brane, which in this case happens to be Y itself, with action given by [10]

$$W = \int_{V} \Omega \wedge \text{Tr}[A \wedge \bar{\partial}A + \frac{2}{3}A \wedge A \wedge A]. \tag{4.2}$$

If we have N branes, A is a holomorphic U(N) gauge connection which can be viewed as a U(N) adjoint valued (0,1) form on Y. For lower dimensional B-branes one obtains the reductions of this action to lower dimensions. For example, for 0-branes all directions of A become scalar⁵.

Here we are interested in the case where the B-branes are two real dimensional (i.e. one complex dimensional) so wrap curves \mathcal{C} in Y. We restrict attention to the case of a single 2-brane and consider the reduction of the holomorphic Chern-Simons theory to \mathcal{C} .

Restricted to \mathcal{C} the tangent space T_Y of the Calabi-Yau Y splits as

$$T_p(Y) = T_p(\mathcal{C}) \oplus N_p(\mathcal{C})$$

where $T_p(\mathcal{C})$ denotes the tangent directions to \mathcal{C} and $N_p(\mathcal{C})$ denotes the normal directions at a point p on \mathcal{C} . Two directions of the gauge field A give two independent sections of the normal bundle $N(\mathcal{C})$, we denote them ϕ^i , i = 1, 2. They should be viewed as deformations of \mathcal{C} in Y.

Since the canonical bundle of Y is trivial, it implies that $\wedge^2 N(\mathcal{C})$ can be identified with $T_{\mathcal{C}}^*$, and the identification is done via contraction with the holomorphic 3-form Ω restricted to \mathcal{C} . In other words, we have the pairing

$$U_z = \Omega_{ijz} \phi^i \wedge \phi^j$$

where z denotes a coordinate system on C. Using this, it is straightforward to write the dimensional reduction of holomorphic Chern-Simons theory on C which is given by

$$W(\mathcal{C}) = \int_{\mathcal{C}} \Omega_{ijz} \phi^i \overline{\partial}_z \phi^j dz d\overline{z}. \tag{4.3}$$

Here we are using a coordinate system on Y where Ω_{ijz} is a constant, as can always be done on a Calabi-Yau three-fold.

⁵ In this case the reduction agrees with the result in [17] for the 0-branes superpotential where the above action becomes $\Omega_{ijk}tr[\Phi^i,\Phi^j]\Phi^k$.

4.2. Another Reformulation of the Superpotential Computation

Note that locally we can write the closed 3-form Ω as

$$\Omega = d\omega$$

in particular $\Omega_{ijz} = \partial_z \omega_{ij} \pm \text{perm}$. Using this and integrating by parts we can rewrite (4.3) as

$$W(\mathcal{C}) = \int_{\mathcal{C}} \omega$$

where here by \mathcal{C} we mean any of the curves arising by deformations of the base curve by the sections of the normal bundle ϕ^i . Note that, even though ω is not globally well defined in general, the above action $W(\mathcal{C})$ is well defined, at least as long as \mathcal{C} has no boundary.

We can now reformulate the superpotential computation in a way which makes contact with another, space-time, viewpoint [21][22], and which we will present in a slightly different form below. This approach has been discussed in the present context in [18].

Consider type IIB superstring on a non-compact Calabi-Yau with a domain wall made of a D5 brane. In x < 0, the 5-brane wraps over the cycle \mathcal{C} and fills the spacetime. At x = 0 it is the three chain D times the 2 + 1 dimensions of spacetime and at x > 0 it wraps over \mathcal{C}_* and fills the spacetime again. Then the BPS tension for this domain wall is given by the "holomorphic volume" of D which is $\int_D \Omega$, and this should correspond to the change in the value of the superpotential from left to right, which is given by $W(\mathcal{C}) - W(\mathcal{C}_*)$. Indeed,

$$W(\mathcal{C}) - W(\mathcal{C}_*) = \int_{\mathcal{C}} \omega - \int_{\mathcal{C}_*} \omega = \int_D \Omega$$
 (4.4)

where D is a 3-chain with $\partial D = \mathcal{C} - \mathcal{C}_*$.

Note that if we consider a family of \mathcal{C} which is holomorphic, then W = 0. One way to see that is to use (4.3) where it is clear that if ϕ^i 's are holomorphic functions of z, i.e. they correspond to a holomorphic deformation of \mathcal{C} , then the superpotential vanishes. Another way to see this is to use (4.4) and note that Ω , which is a (3,0) form restricted to a holomorphic curve \mathcal{C} vanishes. In [18] some non-vanishing superpotentials were obtained by considering a family of curves with obstructed holomorphic deformations, thus giving a non-vanishing W. In our application we find another way W can be non-zero, and that involves considering non-compact C. Fixing the boundary condition at infinity can provide an obstruction for having a holomorphic deformation of C and lead to a non-vanishing superpotential.

In order to do this we will need to apply (4.3) to manifolds C which are non-compact and in these cases, in order to fix the superpotential, we would need to know the boundary conditions on the fields at infinity (which will fix the total derivative ambiguities of the action). This will be discussed later in the context of examples.

4.3. B-brane superpotentials

In this subsection we compute the superpotential for some of the B-branes in non-compact Calabi-Yau three-folds Y considered in section 2 as the mirror of certain A-branes in the mirror non-compact Calabi-Yau.

Consider Calabi-Yau manifold Y given by

$$xz = F(u, v)$$

where F(u, v) = 0 is the equation of complex curve Σ , given by a polynomial in single valued variables e^u , e^v (recall that u, v are cylinder-valued). The appearance of a Riemann surface Σ is familiar from the viewpoint of the N=2 Seiberg-Witten geometry and their realization in terms of geometric engineering of N=2 theories using type II strings propagating on a non-compact Calabi-Yau 3-fold.

We will compute the superpotential for a D-brane wrapping the holomorphic curve C which is one component of the collapsed fiber xz = 0. Concretely, we take

$$C: x = 0 = F(u, v) \quad u = u_* \quad v = v_*,$$

which leaves z arbitrary and we identify it with a coordinate on \mathcal{C} . Thus, \mathcal{C} is non-compact, of complex dimension one, and is parameterized by a point of the Riemann surface Σ , denoted here by u_*, v_* . We will now compute the superpotential as a function of the choice of a point (u, v) on Σ and relate it to Abel-Jacobi map for a 1-form on Σ .

In order to do this, all we have to do is to compute the brane action (4.3) for the configuration of the brane we are considering. Our brane is parameterized by z and the two scalar fields of the theory on the brane can be denoted by $u(z, \overline{z}), v(z, \overline{z})$, which represent its normal deformation inside the Calabi-Yau. We fix the brane so that at infinity it approaches a fixed u_*, v_* , i.e.,

$$u(z,\overline{z}) \to u_*, \quad v(z,\overline{z}) \to v_* \quad \text{as} \quad |z| >> \Lambda$$

for some fixed and large Λ . To begin with we start with the brane for which u, v are identically equal to u_*, v_* for all z. We now want to move the brane to a different value of u, v on the Riemann surface F(u, v) = 0. Since the boundary condition at infinity is fixed this means that we can at most guarantee that $u(z, \overline{z}), v(z, \overline{z})$ for $|z| < \Lambda$ is fixed and equal to (u, v), but that as $|z| \to \infty$ the (u, v) go back to (u_*, v_*) . In fact it is simplest if we consider a rotationally symmetric configuration of u, v on the z plane, so that u, v do not depend on θ , and only depend on r = |z| (see figure 4).

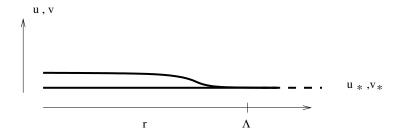


Fig.4: Initial holomorphic brane configuration $(u(r), v(r)) = (u_*, v_*)$ is deformed so that near the origin it is given by (u, v).

In writing the action (4.3), as noted before, in the non-compact case we have to decide which field we keep fixed at infinity. From the A-model side, as discussed before, there is always a natural field corresponding to the variable which measures the size of the disc instantons. Let us say it is v and we thus keep it fixed at infinity (which will of course fix the on-shell value of u, by the condition that F(u,v)=0). The holomorphic three-form on Y is $\Omega=dudv\frac{dz}{z}$, and so using this we can write the action (4.3) as

$$W = \int_{\mathcal{C}} \frac{dz d\overline{z}}{z} u \overline{\partial}_z v$$

(Note that if v were the field fixed at infinity we would have written the action as $-v\overline{\partial}_z u$). For the configuration at hand the $dz/z \sim d\theta$ integral can be readily done and so we are left with the radial integral in the z plane which is given by

$$W(\mathcal{C}) = \int_{v_*}^{v} u dv \to \partial_v W = u. \tag{4.5}$$

The integral can be viewed as the integral of a 1-form on the Riemann surface F(u, v) = 0, and we can view u(v) determined by the condition of being on the Riemann surface. As mentioned before, the superpotential can be viewed as an Abel-Jacobi map associated to the 1-form udv on the Riemann surface F(u, v) = 0 where the position of the brane is labeled by a point on the Riemann surface.

So far we have talked about B-brane configurations which are given by x = 0. We could have done the same for the branes given by z = 0. The only difference between them is given by a change in the sign of the superpotential (because dz/z = -dx/x and so the holomorphic 3-form changes by an overall sign). Note that if we have a copy of both kinds of branes, we can deform the B-branes so that we are no longer on the Riemann surface F(u, v) = 0. Since the superpotential is the addition of these two contributions it vanishes.

5. A-brane Superpotential and Holomorphic Discs

In this section we use the superpotential computation on the B-model mirror to compute holomorphic disc instanton corrections to superpotentials of A-model branes for some of the examples discussed in this paper. The idea is that the disc amplitudes on the A-model side get mapped, by mirror symmetry, to disc amplitudes on the B-model side, which as we discussed in previous section, can be computed explicitly. We will restrict our attention mainly on the A-model Lagrangian submanifolds for which the B-model mirror predicts a non-vanishing disc amplitude. These correspond to the particular class of Lagrangian submanifolds that we discussed in section 3, which end on the 'skeleton' of the toric diagrams.

The right regime for the discussion is the limit where the A-model side is geometric, and that is the large radius limit for the Calabi-Yau. As far as the Lagrangian A-branes are concerned we should also consider the regime of parameters where the discs that bound the branes are large. In this regime of parameters we discussed some non-compact D branes in section 3 and they will serve as our main examples.

As mentioned before, there are strong integrality predictions for the disc amplitudes [6]:

$$W = \sum_{n=0}^{\infty} \sum_{\vec{m}, \vec{k}} \frac{d_{\vec{k}, \vec{m}}}{n^2} q^{n\vec{k}} y^{n\vec{m}}$$
 (5.1)

where $q = e^{-t}$ for the complexified closed string Kahler class t, y is related by exponentiation to the complexified open string Kahler class that measures the volumes of holomorphic discs and $d_{\vec{k},\vec{m}}$ are integers. Below we present some examples. In the first example we give, we recover the corresponding answer predicted in [6] based on a completely different reasoning. In the other examples we obtain more complicated results which as we will discuss below pass the integrality check in a non-trivial way.

5.1.
$$O(-1) \times O(-1)$$
 bundle over \mathbf{P}^1

We consider the small resolution of the conifold given by Q = (1, 1, -1, -1) and the two charges q^1, q^2 denoting the Lagrangian submanifold discussed in section 3. There are two inequivalent "phases" for the Lagrangian submanifolds that we will consider. The two phases are denoted by I and II in figure 2 and we have already discussed, in section 3, how mirror symmetry acts on them. In particular, in terms of the mirror variables $y_2/y_4 = e^u, y_3/y_4 = e^v$ the position of the brane is characterized by u, v subject to

$$0 = e^{-t}e^{v-u} + e^u + e^v + 1.$$

As noted in section 3, in case I the natural variable from the A-model perspective is u, and in the case II the natural variable is v.

Phase I

In this phase u is the physical field of the open string model, which means that it measures the size of a minimal holomorphic disc ending on the Lagrangian submanifold.

Since u is the good variable, the superpotential is $W = -\int v(u)du$, with v(u) determined from the equation of the curve (3.11). For future convenience we use the freedom to shift the imaginary parts of the fields by π , and define new variables $\hat{u} = u + i\pi$, $\hat{v} = v + i\pi$ (as discussed in section 3, the value of the imaginary part is not fixed by mirror symmetry). In terms of shifted variables we have

$$\partial_{\hat{u}}W = -\log\frac{1 - e^{\hat{u}}}{1 - e^{-t - \hat{u}}}.$$

This is in fact the expected answer [6] based on the target space interpretation of topological string amplitudes. To see this it is convenient to factor out $e^{-t-\hat{u}}$ from the denominator which gives

$$W = P(t, \hat{u}) + \sum_{n>0} \frac{e^{n\hat{u}} - e^{n(t+\hat{u})}}{n^2}$$

where P is a finite ambiguous polynomial in t and \hat{u} . This agrees with the result of [6] obtained by completely different means where the two sums were also interpreted in terms of the (multi-coverings) of two primitive discs wrapping the southern and northern hemispheres of the \mathbf{P}^1 and ending on the Lagrangian submanifold.

Phase II

The natural variable for this phase is v which measures the size of the disc passing through the south pole and ending on the Lagrangian submanifold (see figure 5).

⁶ To compare with [6] note that $-\hat{u}$ and $t + \hat{u}$ are the two complexified areas of the two discs.

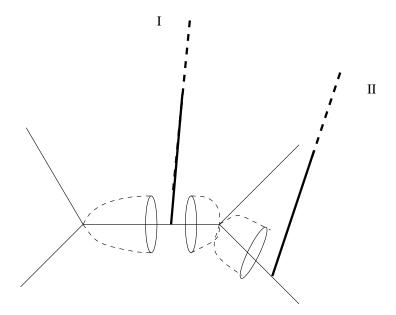


Fig.5: Two phases of the D-brane and some disc instantons.

The superpotential is given by

$$\partial_{\hat{v}}W = \hat{u}(\hat{v}) = \log(\frac{1 - e^{\hat{v}}}{2} + \frac{\sqrt{(1 - e^{\hat{v}})^2 + 4e^{-t + \hat{v}}}}{2}). \tag{5.2}$$

We can expand W around the large radius limit as

$$\partial_{\hat{v}}W = log(1 - e^{\hat{v}}) + \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} C_{k,m} e^{-kt} e^{m\hat{v}}$$

where,

$$C_{k,m} = -\frac{(-1)^k}{k+m}B(k,k,m-k)$$
(5.3)

and where

$$B(a_i) = \frac{(\sum_i a_i)!}{\prod_i a_i!}.$$

To check the integrality properties of this amplitude (5.1) we resum this as

$$\partial_{\hat{v}}W = \sum_{m,k} m d_{k,m} log(1 - q^k e^{m\hat{v}})$$

where $q = e^{-t}$ and $d_{m,k}$ are expected to be integers which label 'the number of primitive discs' wrapping \mathbf{P}^1 k times and wrapping around the S^1 of the Lagrangian submanifold m times. It is quite remarkable that, indeed, doing

the resummation we find that $d_{m,k}$ are integers, as far as we have checked (see table 1). Moreover there are infinitely many non-vanishing integers, (unlike the previous case where there were only two non-trivial integers). It would be quite interesting to verify these numbers directly. Note also that the growth of these numbers is as large as that observed for the primitive rational curves: for discs wrapping \mathbf{P}^1 a fixed number k times and for large wrapping number m on the S^1 , the degeneracies grow like $d_{k,m} \sim m^{2k-2}/(k!)^2$.

m	$d_{0,m}$	$d_{1,m}$	$d_{2,m}$	$d_{3,m}$	$d_{4,m}$	$d_{5,m}$	$d_{6,m}$	
1	1	-1	0	0	0	0	0	
2	0	-1	1	0	0	0	0	
3	0	-1	2	-1	0	0	0	
4	0	-1	4	-5	2	0	0	
5	0	-1	6	-14	14	-5	0	
6	0	-1	9	-31	52	-42	13	
7	0	-1	12	-60	150	-198	132	
8	0	-1	16	-105	360	-693	752	
9	0	-1	20	-171	770	-2002	3114	
10	0	-1	25	-256	1500	-5045	10514	
11	0	-1	30	-390	2730	-11466	30578	
12	0	-1	36	-556	4690	-24024	79420	
13	0	-1	42	-770	7700	-47124	188496	
14	0	-1	49	-1040	12152	-87516	415716	
15	0	-1	56	-1375	18564	-155195	862194	
16	0	-1	64	-1785	27552	-264537	1697472	
:								

Table.1: Holomorphic disc numbers for A-brane on O(-1) + O(-1) over \mathbf{P}^1 in phase II.

5.2. Degeneration of $\mathbf{P}^1 \times \mathbf{P}^1$

The computation of the superpotential in this case can be done from the general formalism we have discussed for the Lagrangian submanifolds ending on the toric skeleton. However, to check integrality properties one has to take into account the closed string mirror map since the quantum corrected areas T_1, T_2 are non-trivial functions of t_1, t_2 . One should, thus, also expect non-trivial analog of mirror map for the boundary variables u, and v. To study this we consider a particular limit of $\mathbf{P}^1 \times \mathbf{P}^1$ where there already is a non-trivial, but relatively simple mirror map. This is the degenerate limit of $\mathbf{P}^1 \times \mathbf{P}^1$ where the size t_2 of the second \mathbf{P}^1 is taken to infinity. In this limit, the equation of the mirror becomes:

$$e^{u} + e^{-t_1 - u} + e^{v} + 1 = 0. (5.4)$$

There are, again, two phases (see figure 6):

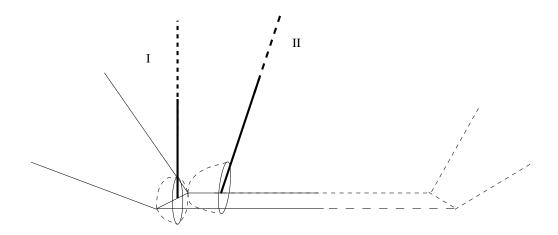


Fig.6: Degenerate limit of $\mathbf{P}^1 \times \mathbf{P}^1$ where the size of second \mathbf{P}^1 is infinite.

Phase I

This is the phase in which the good variable on the curve is u and

$$v(u) = i\pi + \log[e^u + e^{-t_1 - u} + 1].$$

In this case the mirror map gives T_1 in terms of t_1

$$e^{-t_1} = \frac{q}{(1+q)^2},\tag{5.5}$$

where $q = e^{-T_1}$ [23]. It is natural to modify the boundary fields to \hat{u}, \hat{v} such that

$$e^{\hat{u}} = -(1+q)e^{u}$$
, $e^{\hat{v}} = -(1+q)e^{v}$.

To motivate this, note that

$$e^{\hat{u}}e^{T_1/2} = -e^u e^{t_1/2}$$

which is consistent with the fact that when the Lagrangian submanifold intersects the equator of the $\mathbf{P^1}$ both left hand side and the right hand side of the equation should be one. Using this we get

$$\partial_{\hat{u}}W = -\hat{v} = -\log[(1 - e^{\hat{u}})(1 - qe^{-\hat{u}})]. \tag{5.6}$$

Thus, there are again only two primitive disc instantons associated to two hemispheres of the finite size \mathbf{P}^1 , as expected (see figure 6).

Phase II

Solving for \hat{u} in terms of \hat{v} we find

$$\partial_{\hat{v}}W = \hat{u} = \log(\frac{1 + q - e^{\hat{v}}}{2} + \frac{\sqrt{(1 + q - e^{\hat{v}})^2 - 4q}}{2}). \tag{5.7}$$

Expanding this around the large radius limit we find

$$\partial_{\hat{v}}W = \log(1 - e^{\hat{v}}) + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} C_{k,m} q^k e^{m\hat{v}}$$
 (5.8)

where,

$$C_{k,m} = -(-1)^k \frac{B(m,k)}{m+k} - \sum_{n=1}^k (-1)^{k+n} \frac{B(n,n,m,k-n)}{k+n+m}.$$
 (5.9)

Again we find, remarkably, that the $d_{k,m}$ are integers (see table 2), and that the degeneracies of primitive discs grow like $d_{k,m} \sim m^{2k-1}$ for k fixed and m large.

It would be interesting to extend the mirror map computation to the

case where both t_1, t_2 are finite. We are currently investigating this case.

m	$d_{0,m}$	$d_{1,m}$	$d_{2,m}$	$d_{3,m}$	$d_{4,m}$	$d_{5,m}$	$d_{6,m}$	
1	1	1	1	1	1	1	1	
2	0	1	2	4	6	9	12	
3	0	1	4	11	25	49	87	
4	0	1	6	25	76	196	440	
5	0	1	9	49	196	635	1764	
6	0	1	12	87	440	1764	5926	
7	0	1	16	144	900	4356	17424	
8	0	1	20	225	1700	9801	46004	
9	0	1	25	336	3025	20449	111333	
10	0	1	30	484	5110	40080	250488	
11	0	1	36	676	8281	74529	529984	
12	0	1	42	920	12936	132496	1063626	
13	0	1	49	1225	19600	226576	2039184	
14	0	1	56	1600	28896	374544	3755808	
15	0	1	64	2055	41616	600935	6677055	
16	0	1	72	2601	58680	938961	11502216	
:								

Table.2: Holomorphic disc instanton numbers for degeneration of $\mathbf{P}^1 \times \mathbf{P}^1$.

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